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Critical temperature shifts for layered periodic systems in $1/n$ expansion†

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Abstract. A classical n-vector model is considered which is infinitely extended in $d-1$ dimensions and periodic of period b in the dth dimension. Asymptotically for large values of b the critical temperature is expected to obey a scaling law $T_c(b)-T_c(\infty)\sim Ab^{-\lambda}$. We have calculated the shift exponent λ to first order in a $1/n$ expansion for dimensions between three and four. To this order the result is consistent with the assertion $\lambda=\nu_d^{-1}$ where ν_d is the correlation length exponent of the infinitely extended d-dimensional system. From the equivalence between this model and $(d-1)$-dimensional quantum mechanical systems the behaviour of critical lines near the displacive limit ($T_c=0$) can be derived.

1. Introduction

In the past much effort has been spent on calculating critical properties of systems which are finitely extended in one dimension but infinitely extended in the remaining $d-1\equiv\bar{d}$ dimensions (see Fisher 1971, Barber and Fisher 1973, An-Yang and Fisher 1975). The deviations from bulk properties seem to depend crucially on the boundary conditions imposed on the surfaces of the system. The case of periodic boundary conditions which at first sight appears to be rather academic, has recently obtained increased importance since it was discovered that quantum mechanical systems which undergo a second-order phase transition can in certain circumstances be mapped on a classical system with an additional (temperature) dimension in which periodic boundary conditions are imposed (Hertz 1976, Young 1975, Pfeuty 1976, Morf *et al* 1977, Beck and Schäfer 1976, Gerber and Beck 1977). The period b is here given by the reciprocal absolute temperature such that approaching the zero-temperature limit corresponds to letting the width of the corresponding finitely extended system go to infinity. This situation has been reviewed by Fisher (1971) and for the case of quantum effects in the transverse Ising model by Pfeuty (1976). For simplicity we will here stick to the finitely extended system interpretation, keeping in mind that the quantum situations can be described by re-defining the occurring parameters.

One of the basic questions is the behaviour of the critical temperature $T_c(b)$ as the system width b tends to infinity. A general scaling *ansatz* is (Fisher 1971):

$$T_c(b)-T_c(\infty)\sim b^{-\lambda}, \qquad (1.1)$$

with the shift exponent λ.

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On the other hand, crossover scaling ideas predict that critical quantities A like, e.g., the singular part of the specific heat, can be described by a form

$$A_b(T) \sim b^{\omega} X(b^{\phi} t), \tag{1.2}$$

where the function X depends on the scaled variable $b^{\phi}t$ with the crossover exponent ϕ and $t = T - T_c(b)$. There seems to be little doubt that ϕ is equal to the reciprocal of the correlation length exponent ν_d of the infinitely extended system. Regarding the value of λ, however, the situation appears to be rather unclear. For the spherical model Barber and Fisher (1973) obtained the exact result $\lambda = \phi$ for $3 < d < 4$. However, for the Ising model in three dimensions ($d = 3$, $\bar{d} = 2$) Allan (1970) and Fisher (1971) obtained for λ a value which lies somewhere between $\lambda = \nu_d^{-1} = 1{\cdot}56$ and $\lambda = 2$, with emphasis on a high value. This led Pfeuty (1976) to develop in detail the scaling theories for the cases $\lambda \neq \phi$. On the other hand, Morf *et al* (1977) and Beck and Schäfer (1976) proposed the spherical model value $\lambda = \phi = d - 2$, independent of the number of components n of the vector field. In order to clarify the situation a little bit we have decided to calculate the shift in the critical temperature to first order in the $1/n$ expansion (see Abe 1973, Ma 1973) starting from the exact result for the spherical model ($n = \infty$) of Barber and Fisher (1973).

2. Model and basic procedure

Consider a classical n-vector field $\boldsymbol{S}(\boldsymbol{x})$ in a d-dimensional space. We impose periodic boundary conditions in one direction with period $\boldsymbol{b}$, while in the remaining $\bar{d} = d - 1$ dimensions there are no restrictions

$$\boldsymbol{S}(\boldsymbol{x} + \boldsymbol{b}) = \boldsymbol{S}(\boldsymbol{x}). \tag{2.1}$$

The effective Hamiltonian of Landau–Ginsburg–Wilson type has the form

$$\mathcal{H} = \frac{1}{2} \sum_{\boldsymbol{p}} \left(r_0 + \frac{p^2}{\chi(p)} \right) \boldsymbol{S}_{\boldsymbol{p}} \cdot \boldsymbol{S}_{-\boldsymbol{p}} + \frac{u_0}{8} \int \mathrm{d}^d x (\boldsymbol{S}^2(\boldsymbol{x}))^2, \tag{2.2}$$

where $\chi(p)$ is a cutoff function which behaves like $1 + \mathrm{O}(p^2)$ for small wavevectors $\boldsymbol{p}$ while it vanishes fast enough for large $\boldsymbol{p}$ to avoid ultraviolet divergencies in the perturbation expansion. The summation over $\boldsymbol{p}$ is meant to be over a continuous range in $\bar{d}$ dimensions but the values in the additional dimension are restricted to integer multiples of $2\pi/b$. The critical temperature for the extended system ($b = \infty$) is given in terms of the unperturbed critical propagator

$$g(p) = \chi(p)/p^2 \tag{2.3}$$

as an expansion in n^{-1} by

$$T_c\left(\frac{1}{n}\right) = T_c^0 + \frac{1}{n} T_c^1 + \dots, \tag{2.4}$$

where the expressions for T_c^0 and T_c^1 are in convenient units (see Balian 1975)

$$T_c^0 = -2r_0 \left(\int \mathrm{d}^d p \, g(p) \right)^{-1} \tag{2.5}$$

and

$$T_c^1 = -2T_c^0 - \frac{(T_c^0)^2}{2r_0} \int d^dq\, [\Phi(q) + (T_c^0)^{-1}]^{-1}\Psi(q). \tag{2.6}$$

The expressions $\Phi(q)$ and $\Psi(q)$ are given by the integrals

$$\Phi(q) = \tfrac{1}{2} \int d^dp\, g(p) g(|\boldsymbol{q} - \boldsymbol{p}|) \tag{2.7}$$

and

$$\Psi(q) = \int d^dp\, g(p)(g(q)g(|\boldsymbol{q}-\boldsymbol{p}|) + g(p)g(|\boldsymbol{q}-\boldsymbol{p}|) - g(p)g(q)). \tag{2.8}$$

Our aim is to calculate the changes in the expressions (2.5) and (2.6) for T_c^0 and T_c^1 when the system has a finite extension ($b < \infty$) in the asymptotic limit of large b. Technically speaking we look for the changes in the expressions (2.5)–(2.8) when the integral over the dth component of the wavevector is replaced by the discrete sum over the values $2\pi l/b$.

The approach of the sum

$$\textstyle\sum_b = \frac{2\pi}{b} \sum_{l=-\infty}^{\infty} F(2\pi l/b) \tag{2.9}$$

to its limit

$$\textstyle\sum_\infty = \int_{-\infty}^{\infty} dx\, F(x) \tag{2.10}$$

can be calculated by using a continuum version of a method advertised by Barber and Fisher (1973). One considers the Fourier coefficients

$$f(k) = \int_{-\infty}^{\infty} dx\, F(x)\, e^{-ikx}, \qquad f(0) = \textstyle\sum_\infty, \tag{2.11}$$

to obtain for the sum (2.9)

$$\textstyle\sum_b = \sum_\infty + \sum_{k=1}^{\infty} (f(bk) + f(-bk)). \tag{2.12}$$

For T_c^0 it is straightforward to calculate the change with b. One has from (2.5)

$$\delta\left(-\frac{2r_0}{T_c^0}\right) \equiv -2r_0(T_c^0(b)^{-1} - T_c^0(\infty)^{-1})$$
$$= \int d^{\bar d}p \left\{ \frac{2\pi}{b} \sum_l \left[p^2 + \left(\frac{2\pi l}{b}\right)^2\right]^{-1} - \int_{-\infty}^{\infty} dx\, (p^2 + x^2)^{-1} \right\} \tag{2.13}$$

where the cutoff function has been omitted since one has sufficient convergence. Application of (2.12) to the sum yields

$$\delta\left(-\frac{2r_0}{T_c^0}\right) \approx 2\pi \int \frac{d^{\bar d}p}{p(e^{bp} - 1)} = 2\pi S_{\bar d} b^{1-\bar d} \int_0^\infty \frac{x^{\bar d - 2}\, dx}{e^x - 1}, \tag{2.14}$$

where $S_d = 2\pi^{d/2}/\Gamma(d/2)$ is the surface of a d-dimensional unit sphere. It is clear that the cutoff radius Λ is unimportant for $b^{-1} \ll \Lambda$. Corrections will be of the order of $e^{-b\Lambda}$. The integral over x yields $\Gamma(d-1)\zeta(d-1)$ (ζ is Riemann's ζ-function) leading to a value of δT_c^0 which corresponds precisely to the result of Barber and Fisher (1973).

As mentioned we expect the critical temperature shift to obey a scaling law

$$\delta T_c(n) \approx A(n) b^{-\lambda(n)}, \tag{2.15}$$

with the shift exponent $\lambda(n)$. One may assume that $A(n)$ and $\lambda(n)$ have the expansions

$$A(n) = A_0 + A_1 \frac{1}{n} + \ldots, \qquad \lambda(n) = \lambda_0 + \lambda_1 \frac{1}{n} + \ldots, \tag{2.16}$$

where the zeroth order is given from (2.14) by

$$A_0 = 2^{d-2} \pi^{(d-2)/2} \Gamma\left(\frac{d-2}{2}\right) \zeta(d-2) \frac{(T_c^0)^2}{r_0}, \tag{2.17}$$

$$\lambda_0 = d - 2 = \nu_d^{-1}. \tag{2.18}$$

Here the exponent ν_d is associated with the divergence of the correlation length, in the infinitely extended d-dimensional system. The $n = \infty$ model has no transition in two dimensions which is mirrored by the divergence of A_0 as d approaches two due to the pole of $\zeta(x)$ at $x = 1$.

Inserting the expansions (2.16) into the scaling *ansatz* (2.15) yields (see Abe 1973, Kadanoff and Wegner 1971)

$$\delta T_c^1(b) \approx A_1 b^{-\lambda_0} - \lambda_1 A_0 b^{-\lambda_0} \ln b. \tag{2.19}$$

From an evaluation of the asymptotic behaviour of $\delta T_c^1(b)$ for large b one can hence obtain a value for λ_1 by extracting the terms of the form $b^{-\lambda_0} \ln b$, provided the expansions (2.16) exist. The details of this procedure are presented in the next section.

3. Evaluation of T_c^1 for a finite strip

For convenience we start by introducing some notation. The wavevector component corresponding to the finitely extended dimension will carry a subscript zero and the remaining $\bar{d}$-dimensional vector will be barred

$$\boldsymbol{q} = (\bar{\boldsymbol{q}}, q_0). \tag{3.1}$$

Furthermore we write

$$\bar{\int} \mathrm{d}^d q \equiv \int \mathrm{d}^{\bar{d}} \bar{q} \frac{2\pi}{b} \sum_{q_0} \tag{3.2}$$

where q_0 in the sum takes on the values $2\pi l/b$ with integer l. The abbreviation (see (2.7))

$$\delta\Phi(q) \equiv \left(\bar{\int} \mathrm{d}^d p - \int \mathrm{d}^d p\right) \tfrac{1}{2} g(p) g(|\boldsymbol{q} - \boldsymbol{p}|) \tag{3.3}$$

will be used analogously for $\delta\Psi(q)$ (see (2.8)).

We consider the expression (2.6) for T_c^1 in the light of the form (2.19). Keeping in mind our interest in λ_1, one can ignore the changes in T_c^0 which, according to (2.14), will only give contributions to A_1 in (2.19). Furthermore, as was the case in (2.14), only the infrared contribution to the integrals will be important.

The asymptotic forms for $\Phi(q)$ and $\Psi(q)$ for small q for dimensions $d<4$ are (see Balian 1975):

$$\Phi(q)\sim\varphi(d)q^{d-4}, \tag{3.4}$$

$$\Psi(q)\approx\varphi(d)2(4-d)q^{d-6} \tag{3.5}$$

with

$$\varphi(d)=\frac{\pi^2 S_{d-1}}{2^{d-1}\sin[\frac{1}{2}\pi(d-2)]}. \tag{3.6}$$

In the infrared region one may thus neglect $(T_c^0)^{-1}$ in the denominator of the integrand in (2.6) against $\Phi(q)$. Then the quantity of interest is

$$\begin{aligned}-\frac{2r_0}{(T_c^0)^2}\delta T_c^1\approx\tau\equiv&\bar{\int}\mathrm{d}^dq\,\Phi(q)^{-1}\left(\delta\Psi(q)-\frac{\Psi(q)}{\Phi(q)}\delta\Phi(q)\right)\\ \approx&\varphi(d)^{-1}\bar{\int}\mathrm{d}^dq\,q^{2-d}[q^2\,\delta\Psi(q)-2(4-d)\,\delta\Phi(q)].\end{aligned} \tag{3.7}$$

In this expression we have also omitted a contribution of the form

$$\left(\bar{\int}\mathrm{d}^dq-\int\mathrm{d}^dq\right)\Psi(q)/\Phi(q). \tag{3.8}$$

However it is easily seen that this part gives only contributions to A_1 since it has precisely the structure of δT_c^0 (2.14) as is obvious from the expressions (3.4) and (3.5) for $\Phi(q)$ and $\Psi(q)$.

The sums in (3.7) are now evaluated by applying the procedure of equations (2.9)–(2.12). This yields

$$\tau\approx2\pi\varphi(d)^{-1}\sum_s{}'\sum_t\int\mathrm{d}^d\bar{q}\int\mathrm{d}^d\bar{p}\,C(\bar{\boldsymbol{p}},\bar{\boldsymbol{q}},s,t) \tag{3.9}$$

where the sum over s runs over the integer multiplies of b except for the value $s=0$ (which is indicated by the prime on the summation sign) while t runs over all the integer multiples of b. The quantity C is given by the integral

$$C(\bar{\boldsymbol{p}},\bar{\boldsymbol{q}},s,t)=\int_{-\infty}^{\infty}\mathrm{d}q_0\,q^{2-d}\,\mathrm{e}^{-\mathrm{i}q_0t}B(\bar{\boldsymbol{p}},\bar{\boldsymbol{q}},s), \tag{3.10}$$

with

$$B(\bar{\boldsymbol{p}},\boldsymbol{q},s)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{[2\boldsymbol{p}\cdot\boldsymbol{q}+(d-4)p^2]\,\mathrm{e}^{-\mathrm{i}p_0s}\,\mathrm{d}p_0}{(p_0+\mathrm{i}\bar{p})^2(p_0-\mathrm{i}\bar{p})^2(p_0-q_0+\mathrm{i}|\bar{\boldsymbol{p}}-\bar{\boldsymbol{q}}|)(p_0-q_0-\mathrm{i}|\bar{\boldsymbol{p}}-\bar{\boldsymbol{q}}|)}. \tag{3.11}$$

This integral can be treated by residuum integration. Considering first the case $s>0$ one has a contribution B_1 from the double pole at $p_0=-\mathrm{i}\bar{p}$

$$B_1(\bar{\boldsymbol{p}},\boldsymbol{q},s)=\frac{\mathrm{e}^{-\bar{p}s}}{2\bar{p}A_{+-}A_{++}}\left[d-4+\mathrm{i}\frac{q_0}{\bar{p}}+\left(\frac{\bar{\boldsymbol{p}}\cdot\bar{\boldsymbol{q}}}{\bar{p}^2}-\mathrm{i}\frac{q_0}{\bar{p}}\right)\left(1+\bar{p}s+\frac{\mathrm{i}\bar{p}}{A_{+-}}+\frac{\mathrm{i}\bar{p}}{A_{++}}\right)\right], \tag{3.12}$$

with the shorthand notation

$$A_{\epsilon_1,\epsilon_2}=q_0+\mathrm{i}\epsilon_1(\bar{p}+\epsilon_2|\bar{\boldsymbol{p}}-\bar{\boldsymbol{q}}|),\qquad\epsilon_i=+,-. \tag{3.13}$$

The single pole at $p_0 = q_0 - \mathrm{i}|\bar{\boldsymbol{p}} - \bar{\boldsymbol{q}}|$ gives the contribution

$$B_2(\bar{\boldsymbol{p}}, \boldsymbol{q}, s) = \frac{\mathrm{e}^{-|\bar{\boldsymbol{p}}-\bar{\boldsymbol{q}}|s}\,\mathrm{e}^{-\mathrm{i}q_0 s}}{2|\bar{\boldsymbol{p}} - \bar{\boldsymbol{q}}|A_{+-}A_{-+}}\left(d - 3 + \frac{q^2}{A_{+-}A_{-+}}\right). \tag{3.14}$$

For negative values of s we can use the relation

$$B_i(\bar{\boldsymbol{p}}, \boldsymbol{q}, -s) = B_i^*(\bar{\boldsymbol{p}}, \boldsymbol{q}, s). \tag{3.15}$$

Now the expressions (3.12) and (3.14) are inserted into formula (3.10) for C. Anticipating a change of the integration variable $\bar{\boldsymbol{p}} = \bar{\boldsymbol{q}} - \bar{\boldsymbol{p}}'$ and of the summation index $t = t' - s$ in equation (3.9) for the B_2 term, we may replace the quantity C (equation (3.10)) by

$$\bar{C}(\bar{\boldsymbol{p}}, \bar{\boldsymbol{q}}, s, t) = \int_{-\infty}^{\infty} \mathrm{d}q_0\, q^{2-d}\,\mathrm{e}^{-\mathrm{i}q_0 t}(B_1(\bar{\boldsymbol{p}}, \boldsymbol{q}, s) + \mathrm{e}^{\mathrm{i}q_0 s}B_2(\bar{\boldsymbol{q}} - \bar{\boldsymbol{p}}, \boldsymbol{q}, s)), \tag{3.16}$$

to obtain for $s > 0$

$$\bar{C}(\bar{\boldsymbol{p}}, \bar{\boldsymbol{q}}, s, t) = \frac{\mathrm{e}^{-\bar{p}s}}{2\bar{p}}\int_{-\infty}^{\infty} \mathrm{d}q_0\, q^{2-d}\frac{\mathrm{e}^{-\mathrm{i}q_0 t}}{A_{++}A_{+-}}$$
$$\times\left[2d - 7 + \mathrm{i}\frac{q_0}{\bar{p}} + \left(\frac{\bar{\boldsymbol{q}}\cdot\bar{\boldsymbol{p}}}{\bar{p}^2} - \mathrm{i}\frac{q_0}{\bar{p}}\right)\left(1 + \bar{p}s + \frac{\mathrm{i}\bar{p}}{A_{++}} + \frac{\mathrm{i}\bar{p}}{A_{+-}}\right) + \frac{\bar{q}^2 + q_0^2}{A_{++}A_{+-}}\right]. \tag{3.17}$$

It is obvious that $\bar{C}$ effectively depends only on the absolute values $\bar{p}$ and $\bar{q}$ of the vectors $\bar{\boldsymbol{p}}$ and $\bar{\boldsymbol{q}}$ as well as on the angle θ between them. In a somewhat sloppy notation we use the same symbol $\bar{C}$ for this function of $(\bar{p}, \bar{q}, \theta, s, t)$. After a change to the integration variable $k = q_0/\bar{q}$ we obtain

$$\bar{C}(\bar{p}, \bar{q}, \theta, s, t)$$
$$= \frac{\mathrm{e}^{-\bar{p}s}}{2\bar{p}}\bar{q}^{1-d}\left[-\left(2d - 7 + (1 + \bar{p}s)\frac{\bar{q}}{\bar{p}}\cos\theta\right)L_{1,1}^0\right.$$
$$\left. - \bar{q}sL_{1,1}^1 - \cos\theta(L_{1,2}^0 + L_{2,1}^0) - L_{1,2}^1 - L_{2,1}^1 + L_{2,2}^0 - L_{2,2}^2\right], \tag{3.18}$$

where we have introduced the abbreviation

$$L_{m,n}^l(u, w, \bar{q}t) = \int_{-\infty}^{\infty} \mathrm{d}k\,(1 + k^2)^{(2-d)/2}\frac{(-\mathrm{i}k)^l\,\mathrm{e}^{-\mathrm{i}\bar{q}tk}}{(u + 1 - \mathrm{i}k)^m(w - 1 - \mathrm{i}k)^n}. \tag{3.19}$$

In (3.18) u and w have to be set equal to the following expressions:

$$u = \frac{|\bar{\boldsymbol{p}} - \bar{\boldsymbol{q}}|}{\bar{q}} + \frac{\bar{p}}{\bar{q}} - 1, \qquad w = 1 + \frac{\bar{p}}{\bar{q}} - \frac{|\bar{\boldsymbol{p}} - \bar{\boldsymbol{q}}|}{\bar{q}}. \tag{3.20}$$

The important properties of the quantitites $L_{m,n}^l$ are derived in the appendix. Our interest lies in the $\bar{q}$-integral in (3.9). It is easy to see from (3.19) that all the $L_{m,n}^l$ occurring in (3.18) vanish when $\bar{q} \ll \bar{p}$, so that the integral over $\bar{q}$ in (3.9) converges at a lower limit. The singularities at $w = 1$ (which yield singularities in θ near $\theta = \pi/2$ in this limit) are of no importance since they are integrable (the strongest ones occurring in $L_{1,2}^0$ and $L_{2,2}^0$ compensate each other).

Now let us consider the limit $\bar{q} \gg \bar{p}$, which yield $u, w \ll 1$. We first consider the case $t = 0$ and postpone the treatment for $t \neq 0$. From the expansions for small u and w for the $L^{l}_{m,n}(u, w, 0)$ which are listed in the appendix one obtains to leading order from (3.18)

$$\bar{C}(\bar{p}, \bar{q}, \theta, s, 0)$$
$$\approx B(\tfrac{1}{2}, \tfrac{1}{2}d-\tfrac{1}{2})\frac{e^{-\bar{p}s}}{2\bar{p}}\bar{q}^{1-d}\left[\left(2d-7+(1+\bar{p}s)\frac{\bar{q}}{\bar{p}}\cos\theta\right)\left(1+(w-u)\frac{d-1}{d}\right)\right.$$
$$\left.-\bar{q}s\frac{u+w}{d}+\frac{2}{d}+\frac{d-1}{d}+\frac{1}{d}\right]. \tag{3.21}$$

Upon utilising the relations

$$u+w=2\bar{p}/\bar{q}, \qquad w-u\approx 2\bar{p}\cos\theta/\bar{q}, \tag{3.22}$$

and the averages

$$\langle\cos\theta\rangle_\theta=0, \qquad \langle\cos^2\theta\rangle_\theta=(d-1)^{-1}, \tag{3.23}$$

one is left with

$$\langle\bar{C}(\bar{p}, \bar{q}, \theta, s, 0)\rangle_\theta \approx B(\tfrac{1}{2}, \tfrac{1}{2}d-\tfrac{1}{2})\frac{e^{-\bar{p}s}}{2\bar{p}}\bar{q}^{1-d}\left\{\frac{2}{d}(d-2)(d-1)+o\left[\left(\frac{\bar{p}}{\bar{q}}\right)^0\right]\right\}, \tag{3.24}$$

where $B(x, y)$ is the beta function.

In leading order one can hence write

$$\langle\bar{C}(\bar{p}, \bar{q}, \theta, s, 0)\rangle_\theta \approx \frac{e^{-\bar{p}s}}{2\bar{p}}\bar{q}^{1-d}F\left(\frac{\bar{q}}{\bar{p}}\right), \tag{3.25}$$

with

$$F(x)\approx\begin{cases} B(\tfrac{1}{2}, \tfrac{1}{2}d-\tfrac{1}{2})(2/d)(d-2)(d-1), & x\to\infty \\ 0, & x\to 0. \end{cases} \tag{3.26}$$

Now one inserts (3.25) into (3.9) and uses the fact that the form (3.26) effectively sets a lower cutoff at $\bar{q}=\bar{p}$ to the integration. This then yields for the $t=0$ term

$$\tau\approx\sigma 2\pi\int d^d p\sum_s{}'\frac{e^{-\bar{p}s}}{2\bar{p}}\ln\frac{\Lambda}{\bar{p}}, \tag{3.27}$$

where Λ is the natural cutoff in q-space and where σ is given by

$$\sigma=B(\tfrac{1}{2}, \tfrac{1}{2}d-\tfrac{1}{2})\frac{2S_{d-1}}{d\varphi(d)}(d-2)(d-1). \tag{3.28}$$

Remembering (3.15) one can perform the summation over s in (3.27) and change the scale by $x=b\bar{p}$ to obtain

$$\tau\approx\sigma b^{2-d}2\pi S_{d-1}\int_0^\infty\frac{x^{d-2}\,dx}{e^x-1}(\ln b\Lambda-\ln x). \tag{3.29}$$

In this expression one has a term which is precisely of the form we are looking for. Together with (3.7) and (2.14) one can write

$$\delta T_c^1\approx-\delta T_c^0\sigma\ln b, \tag{3.30}$$

where we have omitted terms contributing to A_1 (see (2.19)). From this expression we conclude according to (2.19) that

$$\lambda_1 = \sigma = 2^d \frac{(d-2)(d-1)}{\pi^2 d} \sin\left(\pi \frac{d-2}{2}\right) B(\tfrac{1}{2}, \tfrac{1}{2}d - \tfrac{1}{2}). \tag{3.31}$$

This is our central result.

We have to discuss the terms for non-zero t in (3.9) which we have omitted so far. To this end one has to examine the quantities $L^l_{m,n}(u, w, qt)$ in the limit of small u and w, which is conveniently done in the form (3.19). In this integral the integrand is analytic in a strip $|\mathrm{Im}\,\{k\}| < v = 1 + \mathrm{O}(u, w)$. This gives rise to an exponential decay of L with $\bar{q}$ of the form

$$L^l_{m,n} \sim \mathrm{e}^{-\bar{q}|t|v}. \tag{3.32}$$

Hence by remembering that t is an integer multiple of b we see that this form sets an upper cutoff to the $\bar{q}$ integral of (3.9) at a value b^{-1}. Comparing with (3.29) where for the $t \neq 0$ terms the cutoff Λ would be replaced by b^{-1} we see that no terms of the required form $b^{1-d} \ln b$ can occur. This shows that these terms do not contribute to the exponent correction λ_1.

4. Conclusion

From (2.18) and (3.31) we obtain

$$\lambda = (d-2)\left(1 + \frac{1}{n} 2^d \frac{d-1}{\pi^2 d} \sin[\tfrac{1}{2}\pi(d-2)] B(\tfrac{1}{2}, \tfrac{1}{2}d - \tfrac{1}{2}) + \mathrm{O}(n^{-2})\right). \tag{4.1}$$

This is to be compared with the value of ν_d to the same order. From the paper by Ma (1973) one finds

$$\nu_d = \frac{1}{d-2}\left(1 - \frac{1}{n} 8 \frac{d-1}{d} \frac{2\sin[\tfrac{1}{2}\pi(d-2)]}{\pi(d-2)B(\tfrac{1}{2}d-1, \tfrac{1}{2}d-1)} + \mathrm{O}(n^{-2})\right). \tag{4.2}$$

It is a simple exercise in manipulating beta and gamma functions (see Abramowitz and Stegun 1970) to verify that

$$\lambda = \nu_d^{-1} + \mathrm{O}(n^{-2}). \tag{4.3}$$

This satisfactory result leads one to the conjecture that $\lambda = \nu_d^{-1}$ might be an exact relation in the region $3 < d < 4$. This conjecture together with the assertion $\Phi = \nu_d^{-1}$ leads one to postulate that an extended crossover scaling form applies as well to the present situation as it does for other crossover problems (see e.g. Pfeuty *et al* 1974). Such a form then allows us to determine exponents as seen by letting b^{-1} go to zero at $T_c(\infty)$. A quantity K which has the exponent κ_t when the critical point is approached for the infinite system will diverge like

$$b^{\kappa_t/\nu_d} \tag{4.4}$$

when b is increased to infinity at $T = T_c(\infty)$. As mentioned in the introduction it is straightforward to transfer these results to other situations like displacive limits of structural phase transitions (Morf *et al* 1977, Beck and Schäfer 1976) or to the transverse Ising model (Young 1975, Pfeuty 1976) by simply renaming the parameters. Hence we will not elaborate on that point.

Appendix

We aim to obtain an expansion in u and w valid for small arguments for the integrals (3.19)

$$L^l_{m,n}(u, w, \zeta)=\int_{-\infty}^{\infty} dk\ (1+k^2)^{1-d/2}\frac{(-ik)^l\, e^{-i\zeta k}}{(u+1-ik)^m(w-1-ik)^n}, \tag{A.1}$$

with the condition

$$m+n+d-3-l>0. \tag{A.2}$$

Writing the integrand as a double inverse Mellin transform (Doetsch 1950) one obtains

$$L^l_{m,n}(u, w, \zeta)=\frac{1}{2\pi i}\int_{\tau_0-i\infty}^{\tau_0+i\infty} d\tau\, w^{-\tau}\frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} d\sigma\, u^{-\sigma}$$
$$\times\int_{-\infty}^{\infty} dk\ (-ik)^l(1+k^2)^{1-d/2}W_n(\tau, k)U_m(\sigma, k)\, e^{-i\zeta k}, \tag{A.3}$$

where U and W are given by the Mellin transforms

$$U_m(\sigma, k)=\int_0^{\infty}\frac{u^{\sigma-1}\, du}{(u+1-ik)^m}=B(\sigma, m-\sigma)(1-ik)^{\sigma-m}, \tag{A.4}$$

$$W_n(\tau, k)=\int_0^{\infty}\frac{w^{\tau-1}\, dw}{(w-1-ik)^n}=B(\tau, n-\tau)(-1-ik)^{\tau-n}. \tag{A.5}$$

The integrand paths in (A.3) are determined by the values of σ_0 and τ_0 which have to be chosen such that the integrals (A.4) and (A.5) converge for $\sigma=\sigma_0$ and $\tau=\tau_0$, respectively. This yields the conditions

$$0<\sigma_0<m, \qquad 0<\tau_0<n. \tag{A.6}$$

Inserting (A.4) and (A.5) into (A.3) one obtains

$$L^l_{m,n}(u, w, \zeta)=\frac{1}{2\pi i}\int_{\tau_0-i\infty}^{\tau_0+i\infty} d\tau\, w^{-\tau}B(\tau, n-\tau)$$
$$\times\frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} d\sigma\, u^{-\sigma}B(\sigma, m-\sigma)I_l(m-\sigma, n-\tau, \zeta), \tag{A.7}$$

where

$$I_l(m-\sigma, n-\tau, \zeta)$$
$$=\int_{-\infty}^{\infty} dk\ (ik)^l\, e^{-i\zeta k}(1-ik)^{1+\sigma-m-d/2}(1+ik)^{1+\tau-n-d/2}\exp[i\pi(n-\tau)\operatorname{sgn} k], \tag{A.8}$$

which is analytic in σ and τ provided

$$\sigma+\tau<m+n+d-3-l. \tag{A.9}$$

From (A.7) one can now obtain an asymptotic expansion in ascending powers of u and w by shifting the integration paths of σ and τ to the left, picking up the residues of the poles of the beta functions. This yields in lowest orders

$$L^l_{m,n}(u, w, \zeta) = (-1)^n [I_l(m, n, \zeta) + nwI_l(m, n+1, \zeta) - muI_l(m+1, n, \zeta)] + \mathrm{O}(u^2, uw, w^2). \tag{A.10}$$

For convenience we quote the explicit results for a few values of l, m and n.

$$\begin{aligned}
&L^0_{1,1}(u, w, 0) \approx -B(\tfrac{1}{2}, \tfrac{1}{2}d - \tfrac{1}{2})[1 + (w-u)(d-1)/d],\\
&L^1_{1,1}(u, w, 0) \approx B(\tfrac{1}{2}, \tfrac{1}{2}d - \tfrac{1}{2})(u+w)/d\\
&L^0_{1,2}(u, w, 0) \approx -L^0_{2,1}(u, w, 0) \approx B(\tfrac{1}{2}, \tfrac{1}{2}d - \tfrac{1}{2})(d-1)/d\\
&L^1_{1,2}(u, w, 0) \approx L^1_{2,1}(u, w, 0) \approx -B(\tfrac{1}{2}, \tfrac{1}{2}d - \tfrac{1}{2})/d,\\
&L^0_{2,2}(u, w, 0) \approx B(\tfrac{1}{2}, \tfrac{1}{2}d - \tfrac{1}{2})(d-1)/d,\\
&L^2_{2,2}(u, w, 0) \approx -B(\tfrac{1}{2}, \tfrac{1}{2}d - \tfrac{1}{2})/d.
\end{aligned} \tag{A.11}$$

These are the values which are needed to the quoted order to pass from equation (3.18) to (3.21),

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